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Supermultiplets and relativistic problems: III. The non–relativistic limit for a particle of arbitrary spin in an external field

A Sharma†, M Moshinsky†§ and Yu F Smirnov‡

† Instituto de Física-UNAM, Apartado Postal 20-364, 01000 México, DF Mexico

‡ Instituto de Ciencias Nucleares-UNAM, Apartado Postal 70-543, 04510 México, DF Mexico

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Abstract. In previous papers, with the same series title, an *ab-initio* procedure was developed for deriving a Lorentz invariant equation with arbitrary spins. This equation is linear in the four momentum p_ν, and its coefficients are matrices that can be expressed in terms of ordinary spin and what we called sign spin. In the present paper we consider this equation in an external field A_ν which implies just replacing p_ν by $\Pi_\nu = p_\nu - A_\nu$ and discuss the cases when $A_0 = \frac{1}{2}(r^2/a^2)$ ($a = \sqrt{mc^2/\hbar\Omega}$, Ω being the frequency of the oscillator), $\boldsymbol{A} = 0$ and $A_0 = 0$, $\boldsymbol{A} = \frac{1}{2}(\boldsymbol{r} \times \mathcal{H})$ corresponding respectively to harmonic oscillator potential and a constant magnetic field. By using an appropriate complete set of states, with part of them characterized by the irreps of the chain of groups $\mathrm{SU}(4) \supset \mathrm{SU}_s(2) \otimes \mathrm{SU}_t(2)$ where the subscripts s and t respectively stand for the ordinary and sign spin, the problem can be formulated in a matrix representation whose diagonalization gives the energy spectrum. For simplicity we shall only consider the symmetric representation $\{n\}$ of SU(4) for which $s = t$, and our interest is focussed on the case when the external field is weak, which gives the non-relativistic limit, and where a perturbation analysis can be applied. We show that the expected non-relativistic result can be obtained only when the sign spin projection takes its maximum value, i.e. when all individual states contributing to the final one correspond to positive energies. In the case of constant magnetic field, we obtain the gyromagnetic ratio $1/s$ consistent with other derivations.

1. Introduction

In two previous papers [1, 2] with the same series title (in what follows these will be referred to as I and II) we discussed how a relativistic wave equation of arbitrary spin could be formed from the sum of n free particle Dirac equations in which all the momenta are taken as equal. In I we used the $\boldsymbol{\alpha}_i, \beta,\ i = 1, 2, 3$ formulation of the Dirac equation and in II the $\gamma_\nu,\ \nu = 0, 1, 2, 3$ formulation. However, in both cases our main interest was to show that the matrices mentioned could be decomposed into direct products of ordinary spin matrices and a new type of them that we call sign spin. The problem reduces then to one in terms of the generators of a U(4) group, entirely similar to the one in the spin–isospin theory of nuclear physics and hence the name of supermultiplets in the title. In I we characterized our states by irreps of the $\mathrm{U}(4) \supset \hat{\mathrm{U}}(2) \otimes \check{\mathrm{U}}(2)$ chain of groups in which $\hat{\mathrm{U}}(2)$, $\check{\mathrm{U}}(2)$ are respectively the ones associated with the ordinary and sign spins. In II, where we used the γ notation and proved that our equation of arbitrary spin is Poincaré invariant, we noted that our problem is also invariant under a unitary symplectic subgroup Sp(4) of U(4) and as

§ Member of El Colegio Nacional.

the former is isomorphic to O(5) we could characterize our states by irreps of the canonical chain O(5) $\supset$ O(4) $\supset$ O(3) $\supset$ O(2).

For the integrals of motion associated with a given irrep of O(5) we have, in general, several possibilities of spins and masses for our particle, as shown in II.

In II we discussed only free particles, but in another publication [3] we analysed the effect of an external potential, which for simplicity we took as the harmonic oscillator, on our particle in the $\alpha_{iu}, \beta_u,\ i = 1, 2, 3,\ u = 1, 2, \ldots, n$, formulation. The results for positive energies made sense in the case of a weak potential, i.e. much smaller than mc^2 where m is the lowest mass of our particle, but became less clear when we had a strong potential, i.e. of order mc^2.

This situation led us to discuss in this paper what the correct non-relativistic limit of a particle with arbitrary spins in an external potential is, in order to see from which one we could extrapolate our results as the potential increases. In this paper we start with the γ formulation, so as to have a Poincaré-invariant problem in the free particle case which we called the Bhabha equation [4, 5], and introduce a potential by the minimal extension in which the momentum four vector $cp'_\nu,\ \nu = 0, 1, 2, 3$ is replaced by $cp'_\nu - A'_\nu$, where A'_ν is also a four vector. The prime is used to indicate that all quantities are in CGS units. We shall discuss the problem in a particular frame of reference in which A'_ν takes a simple form, for example $A'_0 = (m\Omega^2 r'^2/2)$, $A'_i = 0;\ i = 1, 2, 3$, the harmonic oscillator potential or $A'_0 = 0$, $A'_i = (e/2)(\boldsymbol{r}' \times \mathcal{H}')_i$, the external constant magnetic field.

2. The Bhabha equation in an external field

From the discussion given in II [2] and the observations in the last paragraph of section 1 above we conclude that the wavefunction of a Bhabha particle in an external field satisfies the equation

$$[\Gamma^\nu(cp'_\nu - A'_\nu) + nmc^2]\psi = 0 \tag{2.1}$$

where, as above, we use CGS units and denote the four momentum p'_ν and its minimal extension A'_ν with a prime, as we wish to reserve the ordinary form of these letters when using appropriate units.

As indicated in equation (2.6) of II Γ^ν is given by

$$\Gamma^\nu = \sum_{r=1}^{n} \gamma_r^\nu \tag{2.2}$$

where γ_r^ν is the direct product of 4×4 matrices

$$\gamma_r^\nu = I \otimes I \cdots \otimes I \otimes \gamma^\nu \otimes I \cdots \otimes I \tag{2.3}$$

with γ^ν in the r position and in it the Pauli matrices $\sigma_i,\ i = 1, 2, 3$ are replaced by σ_{ir}.

As we indicated in II the γ_r^ν can be expressed by direct products involving 2×2 matrices corresponding to the ordinary and sign spins given respectively by $s_{jr}, t_{kr};\ j, k = 1, 2, 3,\ r = 1, 2, \ldots, n$, as well as their appropriate unit matrices $\hat{I}, \check{I}$. From equation (3.6) of II we see that equation (2.1) can then be written as

$$[4iR_{j2}(cp'_j - A'_j) + 2T_3(cp'_0 - A'_0) + nmc^2]\psi = 0 \tag{2.4}$$

where repeated latin indies are summed over their values $j = 1, 2, 3$ and R_{j2}, T_3 are part of the 15 generators of an su(4) Lie algebra given by equation (3.9) of I, i.e.

$$S_i = \sum_{r=1}^{n}(s_{ir} \otimes I) \qquad R_{ij} = \sum_{r=1}^{n}(s_{ir} \otimes t_{jr}) \qquad T_j = \sum_{r=1}^{n}(\hat{I} \otimes t_{jr}). \tag{2.5}$$

As we showed in equations (3.13) and (3.14) of I the ordinary and sign spin part of our states will be characterized by a partition $\{h\}$ of n in at most three numbers $h_1 \geqslant h_2 \geqslant h_3 \geqslant 0$, $h_1 + h_2 + h_3 = n$. In the present paper we shall, for simplicity, limit ourselves to the symmetric partition $h_1 = n,\ h_2 = h_3 = 0$ and denote it by $\{n\}$. Furthermore the ordinary and sign spin states will be characterized by the $\hat{\text{su}}(2) \otimes \check{\text{su}}(2)$ algebras in equation (3.13) of I and their corresponding $\hat{\text{o}}(2) \otimes \check{\text{o}}(2)$ subalgebras so their ket can be denoted as in equation (3.14) of I by

$$|\{n\}s\sigma t\tau\rangle \tag{2.6}$$

where no irrep (s, t) of $\hat{\text{su}}(2) \otimes \check{\text{su}}(2)$ is repeated so the extra quantum number γ in equation (3.14) of I is not required. We note also that for the symmetric representations $\{n\}$ of su(4) the irreps of the subalgebras $\hat{\text{su}}(2)$, $\check{\text{su}}(2)$ are the same [6], i.e. $s = t$ and $\sigma, \tau = s, s-1, \ldots, -s$. Thus in the following we either suppress the t or replace it by s in (2.6).

The orbital part of the states in (2.4) depends on the particular form of the external potential A'_ν and thus will be discussed in the next section.

3. The wave equation for a Bhabha particle in a harmonic oscillator potential

We shall now assume that, in a particular frame of reference, the minimal extension of cp'_ν is an oscillator potential for $\nu = 0$, i.e.

$$A'_0 = \tfrac{1}{2}m\Omega^2 r'^2 \qquad A'_i = 0 \quad i = 1, 2, 3 \tag{3.1}$$

where m is the mass of the particle and Ω the frequency of the oscillator.

The first convenient step for attacking the problem is to divide equation (2.4) by mc^2 and then replace $p'_j,\ j = 1, 2, 3$, by p_j, and r' by r through the relations

$$p'_j = (m\Omega\hbar)^{1/2} p_j \qquad r' = (\hbar/m\Omega)^{1/2} r \tag{3.2}$$

so that equation (2.4) becomes dimensionless.

Furthermore, as A'_ν does not depend on the time, cp'_0 is an integral of motion and in the metric we use, the covariant cp'_0 is equal to the negative of the contravariant one, i.e. $-cp^{'0}$, with the latter being the energy $-E$ of the particle. Thus after carrying out all these replacements we see that equation (2.4) transforms into

$$\left[\frac{4iR_{j2}p_j}{a} - 2T_3\frac{E}{mc^2} + n + 2T_3\frac{1}{2}\frac{r^2}{a^2}\right]\psi = 0 \tag{3.3}$$

where

$$a = \sqrt{\frac{mc^2}{\hbar\Omega}} \tag{3.4}$$

and in the non-relativistic limit $a \gg 1$ or $\hbar\Omega \ll mc^2$.

We note from the definition (2.5) of T_3, that in the basis of the states whose ket is given in (2.6), its matrix representation is diagonal and could be written as

$$\tau\delta_{\tau'\tau}\delta_{s's}\delta_{\sigma'\sigma}. \tag{3.5}$$

If n is odd, τ is semi-integer and, as it cannot take the value 0, the inverse of T_3 exists and is given by (3.5) if τ is replaced by τ^{-1}. If n is even, τ can take the value 0 and thus the inverse remains undefined, but as we can project out this value, as indicated in [7], we can still assume the existence of T_3^{-1} for all values of τ except 0. Furthermore, in equation (3.8) of II we have that the commutator $[T_3, R_{j1}] = iR_{j2}$, so using this result in (3.3) and multiplying equation (3.3) by $(2T_3)^{-1}$ we can write it as

$$(E/mc^2)\psi = \left[(2/a)(R_{j1} - T_3^{-1}R_{j1}T_3)p_j + (2a^2)^{-1}r^2 + n(2T_3)^{-1}\right]\psi \\ \equiv (H/mc^2)\psi \tag{3.6}$$

where the term in the square bracket can now be considered as our Hamiltonian H as its eigenvalue is the energy E, both in units of mc^2.

As equation (3.6) is not soluble exactly we need its matrix formulation with respect to an appropriate basis which we proceed to determine. The ordinary and sign spin part was already obtained in (2.6). For the orbital part we choose a three-dimensional oscillator of unit frequency, as required by relations (3.2), and which we denote by the ket

$$|Nl\mu\rangle = R_{Nl}(r)Y_{l\mu}(\theta, \varphi) \tag{3.7}$$

with $Y_{l\mu}$ being the spherical harmonic and R_{Nl} the radial part.

For the complete ket we couple l and s to the full angular momentum j, as it is an integral of motion because R_{ik} is a first-order Racah tensor [8] in both ordinary and sign spin and it is contracted with the momentum p_i. Thus our ket becomes

$$|N(l,s)jm; \{n\}s\tau\rangle = \sum_{\mu\sigma}\langle l\mu, s\sigma|jm\rangle|Nl\mu\rangle|\{n\}s\sigma s\tau\rangle \equiv |\psi\rangle \tag{3.8}$$

where $\langle\cdot|\cdot\rangle$ is a Clebsch–Gordan coefficient and we take into account that we are dealing with the symmetric irrep $\{n\}$ of su(4) so that $t = s$.

Before evaluating the matrix elements of (H/mc^2) in the square bracket of (3.6), between bras and kets of the form (3.8), it is convenient to express $R_{j1}p_j$ (where the repeated index j is summed over its values $j = 1, 2, 3$) in terms of the spherical components. From the definition of these components for a vector $\boldsymbol{v}$, i.e.

$$v_+ = -(1/\sqrt{2})(v_1 + iv_2) \qquad v_0 = v_3 \qquad v_- = (1/\sqrt{2})(v_1 - iv_2) \tag{3.9}$$

we see that

$$R_{j1}p_j = \sum_q (-1)^q(1/\sqrt{2})(-R_{q+} + R_{q-})p_{-q} \qquad q = +1, 0, -1 \tag{3.10}$$

with the subscript $\pm$ denoting the q-values as $q = \pm 1$.

Using then the standard results of Racah algebras [8], as well as the Wigner–Eckart theorem, we obtain the matrix elements

$$\begin{aligned}
&\langle N'(l',s')jm; \{n\}s'\tau'|(H/mc^2)|N(l,s)jm; \{n\}s\tau\rangle \\
&\quad = \left\{(n/2\tau)\delta_{N'N}\delta_{l'l}\delta_{s's}\delta_{\tau'\tau}\right\} + (1/a)\Big\{2[1 - (\tau/\tau')] \\
&\qquad \times (-1)^{l'+s-j}[(2l'+1)(2s'+1)]^{1/2} \\
&\qquad \times W(ll'ss'; 1j)\langle N'l'\|p\|Nl\rangle\langle\{n\}s's'\|R\|\{n\}ss\rangle \\
&\qquad \times (1/\sqrt{2})[-\langle s\tau, 11|s'\tau'\rangle + \langle s\tau, 1-1|s'\tau'\rangle]\Big\} \\
&\qquad + \left\{(1/2a^2)\langle N'l'\|r^2\|Nl\rangle\delta_{s's}\delta_{\tau'\tau}\delta_{l'l}\right\}
\end{aligned} \tag{3.11}$$

where W and $\langle\cdot|\cdot\rangle$ are respectively the Racah and Clebsch–Gordan coefficients and the reduced matrix elements $\langle N'l'\|p\|Nl\rangle$, $\langle N'l'\|r^2\|Nl\rangle$ are given in [3, equation (5.6)]. It remains for us to discuss the reduced matrix element of $R_{qq'}$, $q = +1, 0, -1$, for the symmetric partition $\{n\}$. It was originally obtained by Ahmed and Sharp [9], but this contained a misprint. We re-derive it in appendix A and obtain

$$\langle\{n\}s's'\|R\|\{n\}ss\rangle = \frac{1}{2}\left\{\delta_{s's+1}\sqrt{\frac{(2s+1)(n/2-s)(n/2+s+2)}{2s+3}}\right.$$
$$\left. + \frac{(n+2)}{2}\delta_{s's} + \delta_{s's-1}\sqrt{\frac{(2s+1)(n/2+s+1)(n/2-s+1)}{2s-1}}\right\}. \tag{3.12}$$

Thus the algebraic expression for all the matrix elements of the operator H/mc^2, given by the square bracket in (3.6) with respect to the states (3.8), is fully determined. We now proceed to analyse its different terms to arrive at its non-relativistic approximation.

We note that in (3.11) there are three curly brackets which correspond to increasing powers of the parameter $1/a$. The first of power 0 in $1/a$ is diagonal in all the quantum numbers, and is given by $n/2\tau$, so it corresponds to different masses (or rest energies) of the Bhabha particle [2], and where τ can take the values $n/2, n/2-1, \ldots, -n/2$. It is the zero-order approximation to our energy (in units of mc^2) and thus we can denote it as

$$\epsilon_0^\tau \equiv n/2\tau. \tag{3.13}$$

Note that in II the letter λ replaces τ.

The second curly bracket is of power 1 in $1/a$. However, We cannot use first-order perturbation theory to evaluate it because, from the Clebsch–Gordan coefficient appearing there, it vanishes if $\tau' = \tau$, whatever the values of the other quantum numbers. Thus we are forced to go to second-order perturbation theory [10] to obtain its contribution to the energy through the evaluation of

$$\epsilon_1^\tau \equiv \frac{1}{a^2}\sum_{\psi'}\frac{\langle\psi|\mathcal{O}|\psi'\rangle\langle\psi'|\mathcal{O}|\psi\rangle}{\epsilon_0^\tau - \epsilon_0^{\tau'}} \tag{3.14}$$

where $|\psi\rangle$ is a shorthand notation for the ket (3.8) while $|\psi'\rangle$ has the same meaning with all the quantum numbers primed except j, m. The operator $\mathcal{O}$ is given by

$$\mathcal{O} = 2[(R_{j1}p_j) - T_3^{-1}(R_{j1}p_j)T_3] \tag{3.15}$$

but as T_3 is diagonal in all quantum numbers and with value τ, using (3.5) we have

$$\epsilon_1^\tau = \frac{4}{a^2}\sum_{\psi'}\left\{\frac{[1-(\tau'/\tau)][1-(\tau/\tau')]}{[n/2\tau)-(n/2\tau')]}\langle\psi|R_{j1}p_j|\psi'\rangle\langle\psi'|R_{j1}p_j|\psi\rangle\right\}$$
$$= \frac{1}{a^2}\frac{8}{n}\sum_{\psi'}\frac{\langle\psi|R_{j1}p_j|\psi'\rangle\langle\psi'|R_{j1}p_j|\psi\rangle}{\tau-\tau'}. \tag{3.16}$$

The matrix element of $\langle\psi'|R_{j1}p_j|\psi\rangle$ is given by the second curly bracket in (3.11) when we suppress the factor $2[1-(\tau/\tau')]$ and thus, because of strong selection rules, the summation over ψ' involves only a few terms and in particular $\tau' = \tau \pm 1$. The analysis is straightforward though laborious and is sketched in appendix B; thus we state only the simple result here:

$$\epsilon_1^\tau = \frac{1}{a^2}\left(\frac{2\tau}{n}\right)\frac{(N+\frac{3}{2})}{2} \qquad \text{for any state } |\psi\rangle \text{ of (3.8).} \tag{3.17}$$

We now turn our attention to the last curly bracket in (3.11) which contains $(1/a)^2$, but where first-order perturbation theory can be applied to obtain

$$\epsilon_2^\tau = \frac{1}{2a^2}\langle\psi|r^2|\psi\rangle = \frac{1}{a^2}\frac{(N+\frac{3}{2})}{2}. \tag{3.18}$$

Thus the total energy up to terms in $(1/a)^2$ is given by

$$\begin{aligned}(E/mc^2) &= \epsilon_0^\tau + \epsilon_1^\tau + \epsilon_2^\tau \\ &= \frac{n}{2\tau} + \frac{1}{a^2}\left(\frac{2\tau}{n}\right)\frac{(N+\frac{3}{2})}{2} + \frac{1}{a^2}\frac{(N+\frac{3}{2})}{2} \\ &= \frac{n}{2\tau} + \frac{1}{a^2}\left(\frac{N+\frac{3}{2}}{2}\right)\left(\frac{2\tau}{n}+1\right).\end{aligned} \tag{3.19}$$

Multiplying both sides of (3.19) by mc^2 and using the definition (3.4) for a, we obtain

$$E = \left(\frac{n}{2\tau}\right)mc^2 + (\hbar\Omega)\left(N+\tfrac{3}{2}\right)\left[\frac{1}{2}\left(\frac{2\tau}{n}+1\right)\right]. \tag{3.20}$$

For a given n the energy E depends only on the total number of quanta N and on the τ eigenvalue of the projection T_3 of the total sign spin.

We note that if τ takes its maximum value $\tau = n/2$ then

$$E - mc^2 = \hbar\Omega(N+\tfrac{3}{2}). \tag{3.21}$$

These would be the eigenvalues of the non-relativistic Hamiltonian

$$\hbar\Omega\tfrac{1}{2}(p^2+r^2) = \frac{p'^2}{2m} + \tfrac{1}{2}m\Omega^2 r'^2$$

which is the one we would expect.

On the other hand, if $\tau \neq n/2$ we would still have an oscillator Hamiltonian but with mass m' and frequency Ω' given by

$$m' = m(n/2\tau) \qquad \Omega' = \Omega\left[\frac{1}{2}\left(\frac{2\tau}{n}+1\right)\right]. \tag{3.22}$$

Thus we see that the correct non-relativistic limit is achieved only when $\tau = n/2$, which implies that all individual sign spin projections involved are pointing upward, i.e. that we are only dealing with positive energy states as is usually assumed for the physical part of the state in relativistic quantum mechanics. Other properties of the results derived in this section will be discussed in the conclusion.

4. The wave equation for a Bhabha particle in a constant magnetic field

We shall assume, in a particular frame of reference, that the minimal extension of cp'_ν is a constant magnetic field which implies we have

$$A'_0 = 0 \qquad A'_j = (e/2)(\boldsymbol{r}'\times\mathcal{H}')_j \quad j=1,2,3 \tag{4.1}$$

in (2.1). The first step for attacking this problem is to divide equation (2.4), in which A'_ν is given by (4.1), by mc^2 to make it dimensionless. Furthermore, as in the previous section $cp'_0 = -cp'^0 = -E$. Besides we replace $p'_j, r'_j, \mathcal{H}'_j$, $j = 1, 2, 3$ through the relations

$$p'_j = mcp_j \qquad r'_j = (\hbar/mc)r_j \qquad \boldsymbol{\mathcal{H}}' = \frac{m^2c^3}{e\hbar}\boldsymbol{\mathcal{H}} \tag{4.2}$$

so that equation (2.4) becomes

$$\left[4iR_{j2}\Pi_j - 2T_3\frac{E}{mc^2} + n\right]\psi = 0 \tag{4.3}$$

in which repeated latin indexes are summed over the values $j = 1, 2, 3$ and Π_j is given by

$$\Pi_j = p_j - \tfrac{1}{2}(\boldsymbol{r} \times \mathcal{H})_j. \tag{4.4}$$

As in the previous section, we now use the commutator $[T_3, R_{j1}] = iR_{j2}$ and the reciprocal of T_3 to write (4.3) in the form

$$\begin{aligned}(E/mc^2)\psi &= \left\{2\left[R_{j1} - T_3^{-1}R_{j1}T_3\right]\left[p_j - \tfrac{1}{2}(\boldsymbol{r} \times \mathcal{H})_j\right] + n(2T_3)^{-1}\right\}\psi \\ &\equiv (H/mc^2)\psi.\end{aligned} \tag{4.5}$$

We select our coordinate frame of reference in such a way that x_3 is in the direction of the magnetic field, and define the creation and annihilation operators in the plane perpendicular to the field as

$$\eta_i = \frac{1}{\sqrt{2}}\left[\left(\frac{\mathcal{H}}{2}\right)^{1/2} x_i - i\left(\frac{\mathcal{H}}{2}\right)^{-1/2} p_i\right] \qquad \xi_i = \frac{1}{\sqrt{2}}\left[\left(\frac{\mathcal{H}}{2}\right)^{1/2} x_i + i\left(\frac{\mathcal{H}}{2}\right)^{-1/2} p_i\right]$$
$$i = 1, 2. \tag{4.6}$$

Furthermore, in spherical components these operators became

$$\eta_\pm = \frac{1}{\sqrt{2}}(\eta_1 \pm i\eta_2) \qquad \xi^\pm = \frac{1}{\sqrt{2}}(\xi_1 \mp i\xi_2). \tag{4.7}$$

The term whose matrix elements are more difficult to determine is $R_{j1}\Pi_j$, so using equations (4.7) and the reasoning in I, we first write it in spherical components of the R_{j1}, as in equation (3.23) of I, i.e.

$$R_{j1}\Pi_j = i\left(\frac{\mathcal{H}}{2}\right)^{1/2}\left[\eta_+(-R_{-+} + R_{--}) + \xi^+(-R_{++} + R_{+-})\right] + \frac{1}{\sqrt{2}}(-R_{0+} + R_{0-})p_3. \tag{4.8}$$

The states with respect to which we want to determine the matrix elements of H are given in equation (3.21) of I by the ket

$$|\mu + \nu - \sigma,\ \nu,\ k\{n\}s\sigma\tau\rangle = \left(\frac{\eta_+^{\mu+\nu-\sigma}\eta_-^{\nu}}{\sqrt{(\mu + \nu - \sigma)!\nu!}}|0\rangle\right)\mathrm{e}^{ikx_3}|\{n\}s\sigma s\tau\rangle \equiv |\psi\rangle \tag{4.9}$$

where $|0\rangle$ is the ground state of the two-dimensional oscillator and, as in section 3, we restrict ourselves to the symmetric partition $\{n\}$ so $s = t$.

Our objective then is to determine the matrix elements

$$\langle\mu + \nu - \sigma', \nu, k\{n\}s'\sigma'\tau'|(H/mc^2)|\mu + \nu - \sigma, \nu, k\{n\}s\sigma\tau\rangle \tag{4.10}$$

where μ is the total angular momentum in the direction of the magnetic field, i.e. $\mu = n_+ - n_- + \sigma$ and n_+, n_- are the number of quanta in $+, -$ directions, i.e. exponents of η_+, η_- in (4.9). The μ does not change in the bra and ket, because it is an integral of motion; the same holds for $\nu \equiv n_-$ and k as well as for the symmetric partition $\{n\}$.

Using the shorthand notation $|s\sigma\tau\rangle$ for the ket (4.9), i.e. suppressing all the integrals of motion, we obtain

$$\langle s'\sigma'\tau'|(H/mc^2)|s\sigma\tau\rangle = \frac{n}{2\tau}\delta_{s's}\delta_{\tau'\tau}\delta_{\sigma'\sigma} + 2(1 - [\tau/\tau'])\langle s'\sigma'\tau'|R_{j1}\Pi_j|s\sigma\tau\rangle. \tag{4.11}$$

Then using the expression (4.8) for $R_{j1}\Pi_j$ we see that we can determine the matrix elements explicitly with the help of the fact that

$$\eta_+|n_+\rangle = \sqrt{n_+ + 1}|n_+ + 1\rangle \qquad \xi^+|n_+\rangle = \sqrt{n_+}|n_+ - 1\rangle. \tag{4.12}$$

We note also the first-order Racah tensor character in both ordinary and sign spin of $R_{q\overline{q}}$, so from the Wigner–Eckart theorem we obtain

$$\langle\{n\}s'\sigma's'\tau'|R_{q\overline{q}}|\{n\}s\sigma s\tau\rangle = \langle s\sigma, 1q|s'\sigma'\rangle\langle s\tau, 1\overline{q}|s'\tau'\rangle\langle\{n\}s's'\|R\|\{n\}ss\rangle \tag{4.13}$$

where $\langle\cdot|\cdot\rangle$ are the Clebsch–Gordan coefficients and the reduced matrix element of R is given explicitly in (3.12).

As our Hamiltonian H is divided by mc^2 we see that to analyse the non-relativistic limit we must assume that the dimensionless $(\mathcal{H}/2)^{1/2}$ and k are much smaller than 1. Thus in equation (4.11) the term of order 0 is given by $n/2\tau$, which corresponds to the different masses (or rest energies in units mc^2) of the Bhabha particle [2] and, as in section 3, we denote it by

$$\epsilon_0^\tau \equiv \frac{n}{2\tau}. \tag{4.14}$$

The second term in (4.11) is of the order $(\mathcal{H}/2)^{1/2}$ or k, both of which we assume much smaller than one, but we cannot consider them in first-order perturbation theory as their matrix elements vanishes if $\tau' = \tau$. We thus have to go to second-order perturbation theory, which from (4.11) becomes

$$\begin{aligned}\epsilon_1^{\tau s\sigma} &\equiv 4\sum_{s'\sigma'\tau'}\frac{[1-(\tau/\tau')][1-(\tau'/\tau)]}{\epsilon_0^\tau - \epsilon_0^{\tau'}}\langle s\sigma\tau|R_{j1}\Pi_j|s'\sigma'\tau'\rangle\langle s'\sigma'\tau'|R_{j1}\Pi_j|s\sigma\tau\rangle\\ &= \frac{8}{n}\sum_{s'\sigma\tau'}\left\{\frac{\langle s\sigma\tau|R_{j1}\Pi_j|s'\sigma'\tau'\rangle\langle s'\sigma'\tau'|R_{j1}\Pi_j|s\sigma\tau\rangle}{\tau-\tau'}\right\}\end{aligned} \tag{4.15}$$

where because of (4.8) we have the selection rules $\tau' = \tau \pm 1;\ \sigma' = \sigma \pm 1, \sigma;\ s' = s \pm 1, s$.

Substituting in (4.15) the value of $R_{j1}\Pi_j$ given in (4.8), one obtains in a straightforward but laborious way, which is sketched in appendix C, the result that

$$\epsilon_1^{\tau s\sigma} = \frac{2\tau}{n}\mathcal{H}(\mu + \nu - \sigma + \tfrac{1}{2}) + \frac{\tau\mathcal{H}\sigma}{ns(s+1)}\left[\left(\frac{n}{2}+1\right)^2 - s(s+1)\right] + \left(\frac{2\tau}{n}\right)\frac{k^2}{2}. \tag{4.16}$$

As $\mu = n_+ - n_- + \sigma, \nu = n_-$, for the total energy we can write

$$(E/mc^2) = \frac{n}{2\tau} + \left(\frac{\mathcal{H}}{2}\right)\left(\frac{2\tau}{n}\right)(2n_+ + 1) + \frac{\tau}{n}k^2 + \frac{\tau\mathcal{H}\sigma}{ns(s+1)}\left[\left(\frac{n}{2}+1\right)^2 - s(s+1)\right]. \tag{4.17}$$

The first term in (4.17) is the rest mass in units of mc^2, the second term comes from the interactions of purely orbital motion with the magnetic field, and is the one we expect non-relativistically if the mass, in units mc^2 of energy, is $n/2\tau$. In addition to being related to τ/n, the last term is related to the spin s and its projection σ. It is this term that gives a non-trivial correction to the non-relativistic result.

As in section 3 the most important case will come when we consider purely positive values of the projection of the sign spin, i.e. when $\tau = n/2$. Then of course $s = n/2$, and our energy in cgs units becomes

$$E - mc^2 = \frac{e\hbar\mathcal{H}'}{mc}(n_+ + \tfrac{1}{2}) + \frac{k'^2}{2m} + \frac{e\hbar}{2mc}\frac{\mathcal{H}'\sigma}{(n/2)}. \tag{4.18}$$

As in this case $n/2 = s$, the gyromagnetic ratio is $1/s$ for the state of maximum sign spin in the symmetric representation $\{n\}$ of su(4). This result is in agreement with other derivations of the gyromagnetic ratio [11–14].

We note, however, that if $\tau \neq n/2$ we obtain a result in (4.17) which is quite different from those we expect in the non-relativistic limit.

5. Conclusion

Our conclusion is that the correct non-relativistic limit corresponds to the maximum projection of the sign spin, i.e. $\tau = n/2$, and that we should extrapolate from this state, variationally and perturbationally, so as to obtain values for strong external fields that make physical sense, i.e. that involve only positive values for the projection of all the sign spins. This suggests that when wishing to obtain the correct spectrum for a relativistic problem, it is convenient to express the γ matrices as direct products of ordinary and sign spin ones. We then select a complete basis on which to give a matrix representation of our relativistic Hamiltonian, but carry out a perturbation calculation on it in which the initial and final states have only the value $+\frac{1}{2}$ for each of the sign spins. This then guarantees that we obtain the positive energy part of our energy spectrum.

Appendix A. Calculation of the reduced matrix elements of R

To find the reduced matrix elements of R for the symmetric partition $\{n\}$, we first start with the basis states which can be obtained by using $\mathrm{SU}(4) \supset \mathrm{S\hat{U}}(2) \otimes \mathrm{S\check{U}}(2)$ chain of groups. We first define $\eta_{\mu i}, \xi^{\mu i}$ as the boson creation and annihilation operators. The index $\mu = 1, 2, 3, 4$ is characterized by spin-sign spin $(\sigma\tau)$ values as follows:

$$\begin{array}{ccccc} \mu & 1 & 2 & 3 & 4 \\ (\sigma\ \tau) & (\frac{1}{2}\ \frac{1}{2}) & (\frac{1}{2}\ -\frac{1}{2}) & (-\frac{1}{2}\ \frac{1}{2}) & (-\frac{1}{2}\ -\frac{1}{2}). \end{array} \tag{A.1}$$

For symmetric partitions the index i takes only one value $i = 1$ which we suppress. The basis states [7] correspond to the product of powers of η_1 and $(\eta_1\eta_4 - \eta_2\eta_3)$. The highest state of the multiplet (ss) is then given by [9]

$$|\{n\}ssss\rangle = \sqrt{\frac{(2s+1)}{(n/2-s)!(n/2+s+1)!}}\,\eta_1^{2s}(\eta_1\eta_4 - \eta_2\eta_3)^{n/2-s}. \tag{A.2}$$

The matrix elements of $R_{q\bar{q}}$ can be calculated by using the following correspondence [15]:

$$R_{11} = \tfrac{1}{2}\mathcal{C}_1^4 \qquad R_{00} = \tfrac{1}{4}\left(\mathcal{C}_1^1 - \mathcal{C}_2^2 - \mathcal{C}_3^3 + \mathcal{C}_4^4\right) \tag{A.3}$$

where $\mathcal{C}_\mu^{\mu'} = \eta_\mu \xi^{\mu'}$ are the SU(4) generators which satisfy

$$\left[\mathcal{C}_\mu^{\mu'}, \mathcal{C}_\nu^{\nu'}\right] = \mathcal{C}_\mu^{\nu'}\delta_\nu^{\mu'} - \mathcal{C}_\nu^{\mu'}\delta_\mu^{\nu'} \qquad \text{since } \left[\xi^{\mu'}, \eta_\mu\right] = \delta_\mu^{\mu'}. \tag{A.4}$$

Using relations (A.2)–(A.4) we obtain

$$R_{11}|\{n\}ssss\rangle = \frac{1}{2}\sqrt{\frac{(2s+1)(n/2+s+2)(n/2-s)}{(2s+3)}}\,|\{n\}s+1, s+1, s+1, s+1\rangle \tag{A.5}$$

$$R_{00}|\{n\}ssss\rangle = \frac{1}{2(s+1)}\sqrt{\frac{(2s+1)(n/2+s+2)(n/2-s)}{(2s+3)}}\,|\{n\}s+1, s, s+1, s\rangle$$

$$+\frac{s(n+2)}{4(s+1)}|\{n\}ssss\rangle. \tag{A.6}$$

We then use relation (4.13), and by noting that $R_{-1-1} = R_{11}^{\dagger}$ we obtain the reduced elements of R as given in (3.12).

Appendix B. Calculation of ϵ_1^τ for the harmonic oscillator

The second-order perturbative energy ϵ_1^τ in $1/a^2$, given in (3.14), can be evaluated using the matrix elements of $R_{ji}p_j$ written in (3.11). By substituting in (3.11) the reduced matrix elements of R shown in (3.12) and those of p given by

$$\langle N'l'\|p\|Nl\rangle = \frac{i}{\sqrt{2}}\left\{\sqrt{\frac{l+1}{2l'+1}}\mathcal{A}^l_{N'N}\delta_{l'l+1} + \sqrt{\frac{l}{2l'+1}}\mathcal{B}^l_{N'N}\delta_{l'l-1}\right\}$$

where

$$\begin{aligned}\mathcal{A}^l_{N'N} &= \sqrt{N+l+3}\delta_{N'N+1} + \sqrt{N-l}\delta_{N'N-1}\\ \mathcal{B}^l_{N'N} &= \sqrt{N-l+2}\delta_{N'N+1} + \sqrt{N+l+1}\delta_{N'N-1}\end{aligned} \tag{B.1}$$

we obtain

$$\begin{aligned}\langle\psi'|R_{ji}p_j|\psi\rangle = {} & \frac{i}{4a}\delta_{s's+1}(-1)^{l'+s-j}\sqrt{(2s+1)(n/2-s)(n/2+s+2)}\\ & \times\left[-\langle s\tau,11|s+1\tau+1\rangle + \langle s\tau,1-1|s+1\tau-1\rangle\right]\\ & \left\{\sqrt{l+1}W(ll+1ss+1;1j)\mathcal{A}^l_{N'N} + \sqrt{l}W(ll-1ss+1;1j)\mathcal{B}^l_{N'N}\right\}\\ & +\frac{i}{8a}\delta_{s's}(-1)^{l'+s-j}\sqrt{(2s+1)}(n+2)\\ & \times\left[-\langle s\tau,11|s\tau+1\rangle + \langle s\tau,1-1|s\tau-1\rangle\right]\\ & \times\left\{\sqrt{l+1}W(ll+1ss;1j)\mathcal{A}^l_{N'N} + \sqrt{l}W(ll-1ss;1j)\mathcal{B}^l_{N'N}\right\}\\ & +\frac{i}{4a}\delta_{s's-1}(-1)^{l'+s-j}\sqrt{(2s+1)(n/2+s+1)(n/2-s+1)}\\ & \times\left[-\langle s\tau,11|s-1\tau+1\rangle + \langle s\tau,1-1|s-1\tau-1\rangle\right]\\ & \left\{\sqrt{l+1}W(ll+1ss-1;1j)\mathcal{A}^l_{N'N} + \sqrt{l}W(ll-1ss-1;1j)\mathcal{B}^l_{N'N}\right\}.\end{aligned} \tag{B.2}$$

We then substitute the explicit expressions for the Clebsch–Gordan and Racah coefficients, and note that

$$\epsilon_1^\tau = \frac{8}{na^2}\sum_{\psi'}\frac{|\langle\psi|R_{j1}p_j|\psi'\rangle|^2}{\tau-\tau'} \tag{B.3}$$

because $R_{j1}p_j$ is a Hermitian matrix. We further note that the terms with $\tau' = \tau - 1$ give positive contributions while those with $\tau' = \tau + 1$ give negative contributions because of the presence of the factor $(\tau - \tau')$ in the denominator of (B.3). Thus we have

$$\epsilon_1^\tau = \frac{\tau(N+3/2)}{4na^2(2l+1)}\left\{\frac{-(n/2-s)(n/2+s+2)}{(s+1)^2(2s+1)}\left[\frac{A}{2l+3}+\frac{B}{2l-1}\right]+\right.$$

$$+\frac{(n/2+1)^2}{s^2(s+1)^2}\left[\frac{C}{2l+3}+\frac{D}{2l-1}\right]$$

$$+\frac{(n/2+s+1)(n/2-s+1)}{s^2(2s+1)}\left[\frac{E}{2l+3}+\frac{F}{2l-1}\right]\Bigg\} \qquad \text{(B.4)}$$

where

$$\begin{aligned} &A=(b+3)(b+2)(d+2)(d+1) \qquad B=(e+2)(e+1)f(f-1)\\ &C=(b+2)(d+1)(f+1)e \qquad D=(b+1)fd(e+1)\\ &E=(f+2)(f+1)e(e-1) \qquad F=(b+1)bd(d-1)\\ &b=j+l+s \qquad d=l+s-j \qquad e=j-l+s \qquad f=j+l-s. \end{aligned} \qquad \text{(B.5)}$$

The algebraic steps to simplify equation (B.4) above are tedious, but give a simple result, as shown in (3.17).

Appendix C. Calculation of $\epsilon_1^{\tau s\sigma}$ for the magnetic field

We first substitute in (4.8) the expressions for η_+ and ξ^+ given in (4.12), and then write $R_{qq'}$ in terms of the reduced matrix elements of R. Further, using equation (4.13) we obtain the following expression for $R_{j1}\Pi_j$:

$$\langle s'\sigma'\tau'|R_{j1}\Pi_j|s\sigma\tau\rangle = \Bigg\{i\sqrt{\frac{\mathcal{H}}{2}}\Bigg[\sqrt{\mu+\nu-\sigma+1}\langle s\sigma,1-1|s'\sigma'\rangle$$

$$+\sqrt{\mu+\nu-\sigma}\langle s\sigma,11|s'\sigma'\rangle\Bigg]+\frac{k}{\sqrt{2}}\langle s\sigma 10|s'\sigma'\rangle\Bigg\}$$

$$\times\left[-\langle s\tau,11|s'\tau'\rangle+\langle s\tau,1-1|s'\tau'\rangle\right]\langle\{n\}s's'\|R\|\{n\}ss\rangle. \qquad \text{(C.1)}$$

We then substitute the Clebsch–Gordan coefficients as well as the reduced matrix elements of R given in (3.12) and simplify. Using arguments similar to those presented in appendix B, we obtain

$$\epsilon_1^{\tau s\sigma}=\frac{\tau}{na^2}\Bigg\{\frac{-(n/2-s)(n/2+s+2)}{(s+1)^2(2s+1)}\Bigg[\frac{\mathcal{H}}{2}\Big(2(\mu+\nu-\sigma)(s^2+\sigma^2+3s+2)$$

$$+(s-\sigma+1)(s-\sigma+2)\Big)+k^2(s-\sigma+1)(s+\sigma+1)\Bigg]$$

$$+\frac{(n/2+1)^2}{s^2(s+1)^2}\Bigg[\frac{\mathcal{H}}{2}\Big(2(\mu+\nu-\sigma)(s^2-\sigma^2+s)+(s+\sigma)(s-\sigma+1)\Big)$$

$$+k^2\sigma^2\Bigg]+\frac{(n/2+s+1)(n/2-s+1)}{s^2(2s+1)}\Bigg[\frac{\mathcal{H}}{2}\Big(2(\mu+\nu-\sigma)(s^2+\sigma^2-s)$$

$$+(s+\sigma)(s+\sigma-1)\Big)+k^2(s^2-\sigma^2)\Bigg]\Bigg\} \qquad \text{(C.2)}$$

which can be further simplified to obtain the final result shown in (4.16).

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